

The Coulomb Phase Revisited

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Abstract

Motivated by the forthcoming data from the E950 experiment at BNL for small angle polarized proton-carbon scattering we revisit the problem of Coulomb phase. The approximation usually used, with the momentum transfer squared (q^2) small compared to the inverse elastic slope, is not justified within the kinematics of the E950 experiment. We go beyond this approximation and derive a new rather simple expression which recovers the formula of Bethe at small q^2 , but is valid at any momentum transfer.

The forthcoming precise data for small angle polarized proton-carbon scattering measured in the E950 experiment at BNL raise again the problem of Coulomb phase [1]. The experiment covers the range of momentum transfer up to $q^2 = 0.05 \text{ GeV}^2$ where the product $Bq^2 = 3$ ($B \approx 60 \text{ GeV}^2$ is the elastic slope). At the same time the early calculations of the Coulomb phase neglected terms of the order of Bq^2 [1, 2]. The next-to-leading corrections of the order of $q^2 \ln(q^2)$ were calculated by Cahn [3]. However, next-to-next-to-leading order terms might be also important at higher q^2 .

The asymmetry of polarized proton-nucleus scattering arising from Coulomb-nuclear interference [4], the main goal of experiment E950, can be essentially affected by the Coulomb phase, especially if the hadronic spin-flip amplitude has a real part [5]. This situation motivates us to attack once again the problem of the Coulomb phase to reveal the corrections of the order of q^2 and higher. This has recently been done numerically for pp scattering in [6].

We demonstrate that one can arrive at a rather simple analytical expression Eq. (34) for the Coulomb phase which is valid at any q^2 , provided that the electromagnetic formfactor and nuclear amplitude have Gaussian dependence on the momentum transfer q . This assumption is well justified for light and medium heavy nuclei. For a proton target the Gaussian formfactor falls off too steeply at $q^2 > \Lambda^2$, where $\Lambda^2 = 0.71 \text{ GeV}^2$ is the parameter of the dipole parameterization. However, the Coulomb phase matters only in the Coulomb-nuclear interference region $q^2 \ll 0.1 \text{ GeV}^2$ where the Gaussian parameterization is precise.

If the two slopes of q^2 dependence for electromagnetic and hadronic amplitude coincide the expression for the Coulomb phase becomes especially simple as is given by Eq. (35).

The elastic scattering amplitude can be represented as,

$$f(q) = f_C(q) + f_{NC}(q) , \quad (1)$$

where the first and second terms correspond to the first (a) and sum of the second and third (b,c) graphs depicted in Fig. 1, respectively. Of course such a classification is conventional

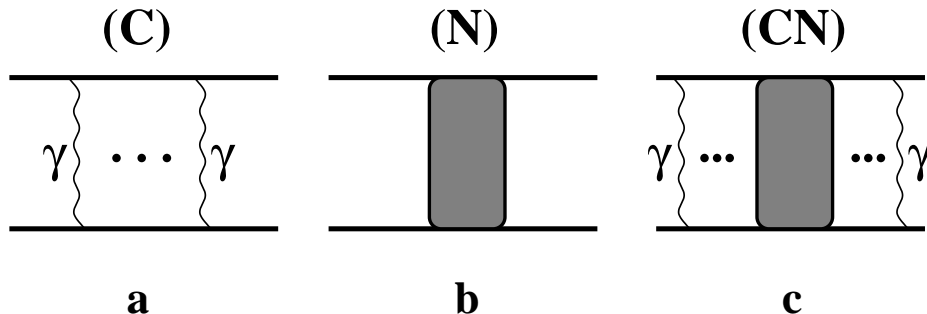


Figure 1: Three types of interaction: pure Coulomb (a), nuclear (b) and nuclear-Coulomb (c).

and we make it for convenience.

The amplitudes in Eq. (1) read,

$$f_C(q) = \frac{i}{2\pi} \int d^2b e^{i\vec{q}\cdot\vec{b}} \left(1 - e^{i\chi_C(b)} \right) , \quad (2)$$

$$f_{NC}(q) = \frac{i}{2\pi} \int d^2b e^{i\vec{q}\vec{b}} e^{i\chi_C(b)} \gamma_N(b) , \quad (3)$$

where

$$\gamma_N(b) = \frac{i}{2\pi} \int d^2q e^{-i\vec{q}\vec{b}} f_N(q) . \quad (4)$$

The amplitudes Eqs. (1) - (4) are normalized according to the conditions,

$$\begin{aligned} \frac{d\sigma}{dq^2} &= |f(q)|^2 \\ \sigma_{tot} &= 4\pi \text{Im } f_N(q=0) . \end{aligned} \quad (5)$$

In Eqs. (1) - (5) $f_C(q)$ and $f_N(q)$ are the net Coulomb (long-range) and nuclear (proton-nucleus) amplitudes (Figs. 1a and b respectively), and $f_{NC}(q)$ includes the effects of strong interaction (Fig. 1b) and Coulomb-nuclear interference (Fig. 1c).

The Coulomb phase $\chi_C(b)$ in Eqs. (1)-(4) is a Fourier transform of the Coulomb part of the amplitude calculated in the Born approximation,

$$\chi_C(b) = \frac{1}{2\pi} \int d^2q e^{-i\vec{q}\vec{b}} f_C^{(B)}(q) ; \quad (6)$$

$$f_C^{(B)}(q) = -\frac{2\alpha Z_1 Z_2}{q^2 + \lambda^2} S(q^2) ; \quad (7)$$

$$S(q^2) = F_{em}^{(1)}(q^2) F_{em}^{(2)}(q^2) , \quad (8)$$

where $F_{em}^{(1,2)}(q^2)$ are the electromagnetic formfactors of the colliding particles (nuclei), and $Z_{1,2}$ are their charges. In order to keep the integrals finite we give to the photon a small mass λ which will disappear from the final expressions in the limit $\lambda \rightarrow 0$.

On the contrary to the Born approximation, the full Coulomb amplitude including all the multi-photon exchanges is complex, *i.e.* has a phase $\Phi_C(q)$,

$$f_C(q) = -\text{sign}(Z_1 Z_2) |f_C(q)| e^{i\Phi_C(q)} , \quad (9)$$

where

$$|f_C(q)| = |f_C^{(B)}(q)| \left[1 + O((\alpha Z_1 Z_2)^2) \right] . \quad (10)$$

For the sake of simplicity we restrict our consideration to the case of $\alpha|Z_1 Z_2| \ll 1$ (appropriate for pC) and will neglect the higher order corrections. In this approximation,

$$\begin{aligned} \Phi_C(q) &= \frac{1}{2} \frac{\int d^2b |\chi_C(b)|^2 e^{i\vec{q}\vec{b}}}{\int d^2b \chi_C(b) e^{i\vec{q}\vec{b}}} \\ &= \frac{1}{2\pi f_C^{(B)}(q)} \int d^2q_1 d^2q_2 f_C^{(B)}(q_1) f_C^{(B)}(q_2) \delta(\vec{q} - \vec{q}_1 - \vec{q}_2) . \end{aligned} \quad (11)$$

With the same accuracy the amplitude $f_{NC}(q)$ in Eq. (3) can be represented in the form,

$$f_{NC}(q) = f_N(q) e^{i\Phi_{NC}(q)} + O((\alpha Z_1 Z_2)^2) , \quad (12)$$

where

$$f_{NC}(q) \Phi_{NC}(q) = \frac{1}{2\pi} \int d^2q_1 d^2q_2 \delta(\vec{q} - \vec{q}_1 - \vec{q}_2) . \quad (13)$$

Our results for Φ_C , Φ_{NC} and $\Delta\Phi = \Phi_C - \Phi_{NC}$ look rather simple if the q -dependence of the formfactors in Eq. (3) is Gaussian,

$$S(q^2) = e^{-a q^2}, \quad a = \frac{1}{6} (\langle r^2 \rangle_1 + \langle r^2 \rangle_2), \quad (14)$$

$$f_N(q) = f_N(0) e^{-b q^2}, \quad b = B/2, \quad (15)$$

where B is the slope of the hadronic differential cross section.

In this case the phase Eq. (13) takes the form,

$$\Phi_{NC}(q) = -\frac{\alpha Z_1 Z_2}{\pi} \int \frac{d^2 q_1}{q_1^2 + \lambda^2} \exp \left[-(a+b) q_1^2 + 2b \vec{q}_1 \cdot \vec{q} \right]. \quad (16)$$

This integration can be performed analytically replacing

$$\frac{1}{q_1^2 + \lambda^2} = \int_0^\infty dt \exp \left[-t (q_1^2 + \lambda^2) \right]. \quad (17)$$

Then the phase Eq. (16) takes the form,

$$\Phi_{NC}(q) = -\frac{\alpha Z_1 Z_2}{\pi} \int_1^\infty \frac{du}{u} \exp \left[\frac{z}{u} - v u + v \right], \quad (18)$$

where

$$\begin{aligned} z &= \frac{b^2 q^2}{a+b}, \\ v &= \lambda^2 (a+b). \end{aligned} \quad (19)$$

Further, expanding the exponential we get,

$$\Phi_{NC}(q) = -\frac{\alpha Z_1 Z_2}{\pi} \sum_{k=0}^\infty \frac{z^k}{k!} E_{k+1}(v) e^v, \quad (20)$$

where

$$\begin{aligned} E_{k+1}(v) &= \int_1^\infty \frac{du}{u^{k+1}} e^{-kv}, \\ E_{k+1}(v)|_{v \rightarrow 0} &= \frac{1}{k}, \quad k > 0, \\ E_1(v)|_{v \rightarrow 0} &= -\gamma - \ln(v). \end{aligned} \quad (21)$$

Here $\gamma = 0.5772$ is the Euler constant.

Thus at $v \ll 1$ we arrive at,

$$\Phi_{NC}(q) = \alpha Z_1 Z_2 \left[2\gamma + \ln(vz) - \text{Ei}(z) \right] + O(v), \quad (22)$$

where $\text{Ei}(z) = -E_1(-z)$ is the integral exponential function.

Coming back to the phase (11) we represent the integral in the numerator in the *r.h.s.* as,

$$\begin{aligned} I_c(q) &= \frac{1}{4\pi} \int d^2 q_1 d^2 q_2 f_C^{(B)}(q_1) f_C^{(B)}(q_2) \delta(\vec{q} - \vec{q}_1 - \vec{q}_2) \\ &= \frac{1}{16\pi} \int d^2 \Delta f_C^{(B)}\left(\frac{\vec{q} + \vec{\Delta}}{2}\right) f_C^{(B)}\left(\frac{\vec{q} - \vec{\Delta}}{2}\right). \end{aligned} \quad (23)$$

Since the product

$$S\left(\frac{\vec{q} + \vec{\Delta}}{2}\right) S\left(\frac{\vec{q} - \vec{\Delta}}{2}\right) = \exp\left[-\frac{a}{2}(q^2 + \Delta^2)\right] \quad (24)$$

is independent of the angle ϕ between the vectors \vec{q} and Δ ($d^2 \Delta = d\Delta^2 d\phi/2$), one can perform integration over ϕ in (23),

$$\begin{aligned} I_c(q) &= \frac{2(\alpha Z_1 Z_2)^2}{\pi} \int_0^\infty d\Delta^2 \int_0^{2\pi} d\phi \exp\left[-\frac{a}{2}(q^2 + \Delta^2)\right] \\ &\times \left[(q^2 + 2q\Delta \cos \phi + \Delta^2 + 4\lambda^2)(q^2 - 2q\Delta \cos \phi + \Delta^2 + 4\lambda^2)\right]^{-1}. \end{aligned} \quad (25)$$

Then, it can be represented as,

$$I_c(q) = 2(\alpha Z_1 Z_2)^2 \int_0^\infty d\Delta^2 \exp\left[-\frac{a}{2}(q^2 + \Delta^2)\right] \frac{\partial \Psi(\Delta^2, q^2, \lambda^2)}{\partial \Delta^2}, \quad (26)$$

where

$$\Psi(\Delta^2, q^2, \lambda^2) = \frac{1}{\sqrt{q^2(q^2 + \lambda^2)}} \ln \left[\sqrt{\left(u - \frac{1}{2}\right)^2 + \frac{\lambda^2}{q^2}} + u - \frac{1}{2} \right], \quad (27)$$

$$u = \frac{q^2 + 4\lambda^2}{q^2 + \Delta^2 + 4\lambda^2}. \quad (28)$$

In the limit $\lambda \rightarrow 0$ function $\Psi(\Delta^2, q^2, \lambda^2)$ can be expanded dependent on the relation between Δ^2 and q^2 ,

$$\begin{aligned} \Psi(\Delta^2, q^2, \lambda^2)|_{\Delta^2 < q^2} &= \frac{1}{q^2} \ln \left(\frac{q^2 - \Delta^2}{q^2 + \Delta^2} \right) + O\left(\frac{\lambda^2}{q^2}\right), \\ \Psi(\Delta^2, q^2, \lambda^2)|_{\Delta^2 > q^2} &= \frac{1}{q^2} \left[\ln \left(\frac{\lambda^2}{q^2} \right) - \ln \left(\frac{\Delta^2 - q^2}{\Delta^2 + q^2} \right) \right] + O\left(\frac{\lambda^2}{q^2}\right), \end{aligned} \quad (29)$$

Performing integration in (26) in parts and taking into account relation (29) we arrive at,

$$\begin{aligned} I_c(q) &\equiv f_C^{(B)}(q) \Phi_C(q) = -\alpha Z_1 Z_2 f_C^{(B)}(q) \left[\ln \left(\frac{q^2}{\lambda^2} \right) + w \int_{-1}^0 dt e^{-wt} \ln \left| \frac{2+t}{t} \right| \right. \\ &\quad \left. - w \int_0^\infty dt e^{-wt} \ln \left(\frac{2+t}{t} \right) \right] + O\left(\frac{\lambda^2}{q^2}\right), \end{aligned} \quad (30)$$

where

$$w = \frac{a q^2}{2} . \quad (31)$$

The integrals contained in this expression can be represented in terms of the integral exponential function,

$$\begin{aligned} w \int_{-1}^0 dt e^{-wt} \ln \left| \frac{2+t}{t} \right| &= \text{Ei}(w) - \gamma - \ln(2w) + e^{2w} \left(\text{E}_1(w) - \text{E}_1(2w) \right) , \\ w \int_0^\infty dt e^{-wt} \ln \left(\frac{2+t}{t} \right) &= \gamma + \ln(2w) + e^{2w} \text{E}_1(2w) . \end{aligned} \quad (32)$$

Eventually we arrive at the following expression for the phase $\Phi_C(q)$,

$$\Phi_C(q) = \alpha Z_1 Z_2 \left\{ 2 \ln(2w) - \ln \left(\frac{q^2}{\lambda^2} \right) + 2\gamma - \text{Ei}(w) + e^{2w} \left[2 \text{E}_1(2w) - \text{E}_1(w) \right] \right\} . \quad (33)$$

Apparently, the common phase factor for the terms in the elastic amplitude (1) is unobserved, only the difference $\Delta\Phi = \Phi_C - \Phi_{NC}$, the so called Coulomb phase, matters,

$$\Delta\Phi(q) = \alpha Z_1 Z_2 \left\{ \ln \left(\frac{a^2}{b^2} \right) + \text{Ei}(z) - \text{Ei}(w) + e^{2w} \left[2 \text{E}_1(2w) - \text{E}_1(w) \right] \right\} . \quad (34)$$

Note that all the divergences and the photon mass λ have cancelled.

If the slopes of the Coulomb and hadronic formfactors coincide, $a = b$ ($z = aq^2/2 = w$), the expression (34) takes an especially simple form,

$$\Delta\Phi(q) = \alpha Z_1 Z_2 e^{2w} \left[2 \text{E}_1(2w) - \text{E}_1(w) \right] . \quad (35)$$

This is a good approximation for nuclei where the slopes of the electromagnetic and hadronic formfactors are mainly determined by the nuclear radius. For a proton target the electromagnetic slope $a = 4/\Lambda^2 = 5.6 \text{ GeV}^{-2}$. The slope of the hadronic amplitude b is approximately equal to a at medium-high energies. However, it increases with energy and at very high energies, in particular at the Tevatron, b substantially exceeds a and one should use the exact expression (34).

To see the difference of our results from the previous calculations we show $\Delta\Phi(q^2)$ by the solid curve in Fig. 2. We performed calculations for proton-carbon elastic scattering using (35) with $a = 30 \text{ GeV}^{-2}$. Dotted and dashed curves correspond to the formulas of Bethe [1] and Cahn [3] (with $B = 8/\Lambda^2 = 2a$) respectively. At small q^2 dominated by the leading $\sim \ln(q^2)$ term the results of Cahn and ours coincide and deviate from the Bethe's due to the next-to-leading corrections $\sim q^2 \ln(q^2)$. At higher q^2 the next-to-next-to-leading corrections $\sim q^2$ become important and all three curves are quite different. This is not surprising since it is not legitimate to use the results of [1, 3] at so large q^2 .

Note that the eikonal approximation we use is subject to inelastic corrections. In addition to the ordinary inelastic corrections to the hadronic amplitude which we assume to be already included, excitation of inelastic states N^* between the electromagnetic and hadronic

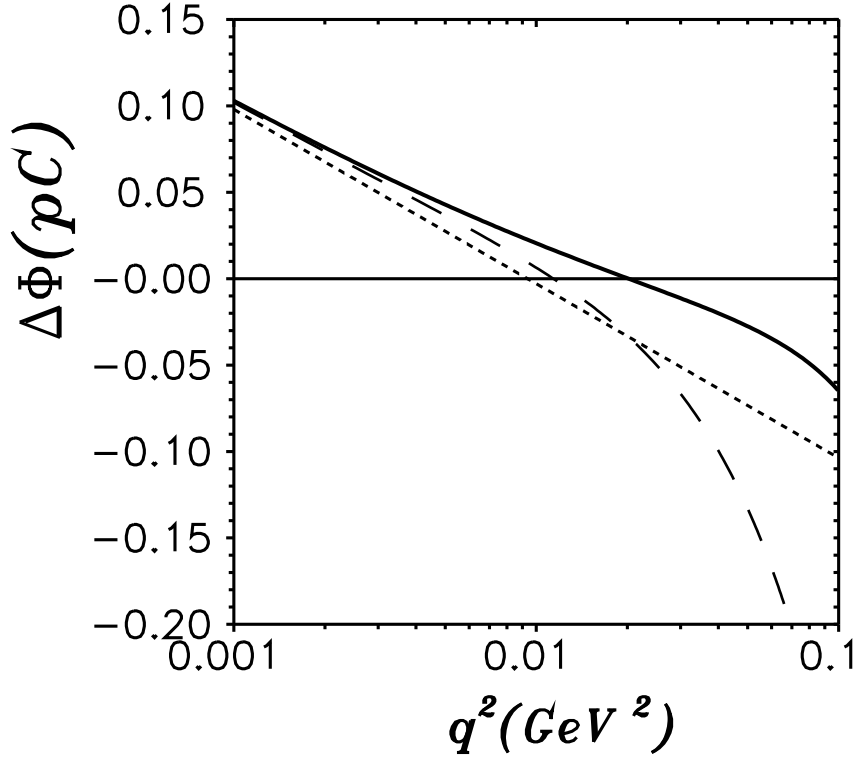


Figure 2: *The Coulomb phase as function of q^2 . The dotted, dashed and solid curves corresponds to the formulas of Bethe [1], Cahn [3], and present calculations, respectively.*

vertices in Fig. 1c can affect our results. Nevertheless, it turns out that the correction to the Coulomb phase is quite small. Indeed, the $N \rightarrow N^*$ amplitude contains an extra factor q compared to the elastic $N \rightarrow N$ one. At small $q \rightarrow 0$ this vertex squared cancels $1/q^2$ in the photon propagator and this correction looks similar to the ordinary inelastic one. We only have to replace factor A^2/R_A^2 in the hadronic inelastic correction to $A Z \alpha/R_A^2$. Then, αZ is the common factor for the Coulomb phase, and the relative correction becomes $1/[(m_{N^*}^2 - m_N^2)R_A^2]$. This is a small correction, about 10^{-2} , to the constant and $q^2 R_A^2$ terms in the phase.

Concluding, we derived a new expression (34) for the Coulomb phase which is valid at any q^2 provided that the electromagnetic formfactor and the hadronic amplitude have Gaussian dependences on q . If their slopes are equal the Coulomb phase takes a very simple form (35).

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